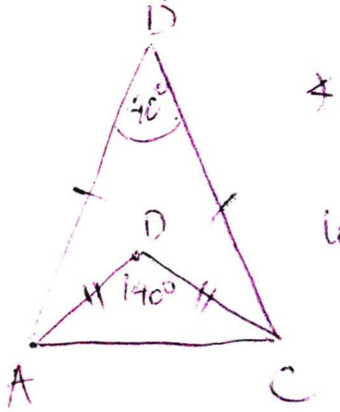
Math Circle 10/3/09

Problem Session in Geometry

1)

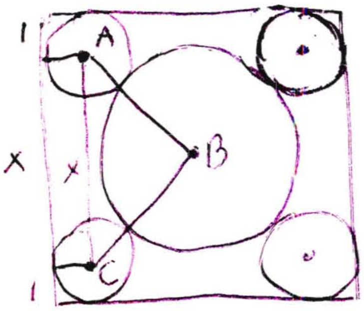


$\measuredangle BAD = \measuredangle BAC - \measuredangle DAC$

$= 70^\circ - 20^\circ = 50^\circ$

We used $\measuredangle BAC = \frac{180^\circ - 40^\circ}{2}$, $\measuredangle DAC = \frac{180^\circ - 140^\circ}{2} = 20^\circ$

2)



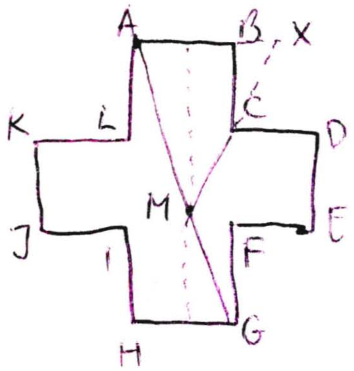
From $\triangle ABC$ which is right isosceles with leg 3

we get that $AC = x = 3\sqrt{2}$

Side of the square: $3\sqrt{2}+2$

Area: $(3\sqrt{2}+2)^2 = 22+12\sqrt{2}$.

3) $\triangle AMH \sim \triangle GMC$ and $\frac{AH}{GC} = \frac{3}{2}$ Therefore $\frac{AM}{MG} = \frac{3}{2}$.



$\triangle AMX \sim \triangle GMH$ and their ratio is also $3:2$.

Therefore:

1) Their altitudes are $\frac{3}{5}\cdot 12$ and $\frac{2}{5}\cdot 12$

2) $AX = 6$ so $BX = 2$

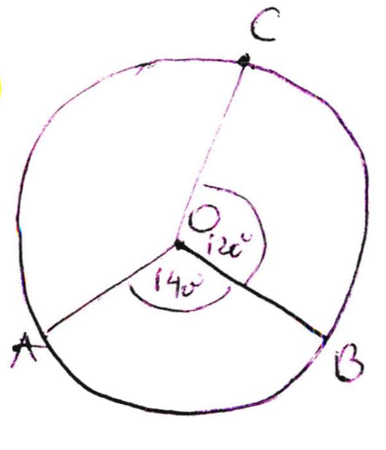
$A_{\triangle HGM} = \frac{1}{2}\cdot 4\cdot \frac{24}{5} = \frac{48}{5}$

$A_{\triangle AXM} = \frac{1}{2}\cdot 6\cdot \frac{36}{5} = \frac{108}{5}$

$A_{\triangle BXC} = \frac{1}{2}\cdot 2\cdot 4 = 4$

$A_{\square ABCM} = \frac{108}{5} - 4 = \frac{88}{5}$.

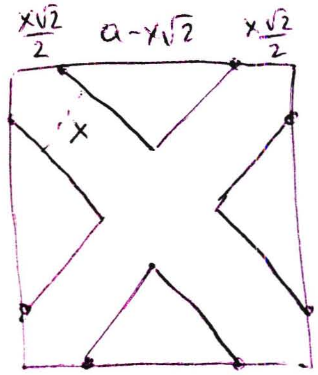
4)



$\measuredangle AOC = 360^\circ - (120^\circ + 140^\circ) = 100^\circ$

$\measuredangle ABC = \frac{1}{2}\measuredangle AOC = 50^\circ$

5)

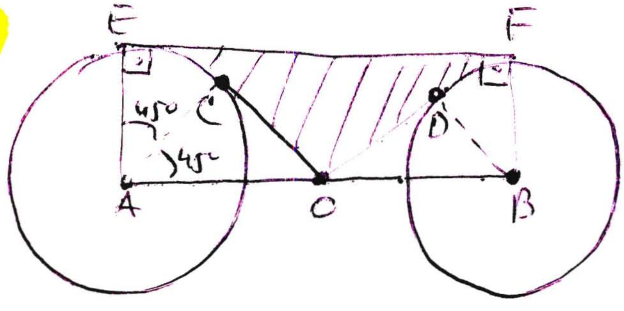


$$A_\Delta = \frac{(a-x\sqrt2)^2}{4}$$

We also know $4A_\Delta = \frac{a^2}{2}$. Comparing this we have:

$$2(a-x\sqrt2)^2 = a^2$$
$$\sqrt2(a-x\sqrt2) = a$$
$$\sqrt2 a - a = 2x \longrightarrow \frac{a}{x} = \frac{2}{\sqrt2-1} = 2\sqrt2+2.$$

6)

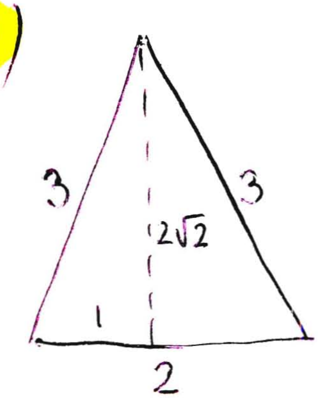


Since ΔAOC is a right angle triangle with hypotenuse $2\sqrt2$ and leg 2, $\measuredangle OAC = 45°$. Therefore $\measuredangle CAE = 90° - 45° = 45°$. We need to subtract from the area of the rectangle ABFE which is $8\sqrt2$, two triangles and two 45° sectors, so we have: $8\sqrt2 - 4 - \frac{\pi\cdot 4}{4} =$

$$= 8\sqrt2 - 4 - \pi.$$

7) $A_{\Delta ABC} = \frac{1}{2}\overline{BC}\cdot h = 1 \longrightarrow$ h is fixed. So points A are at a fixed distance from line $\overline{BC}$, so they belong to one of the two parallel lines at that distance from BC.

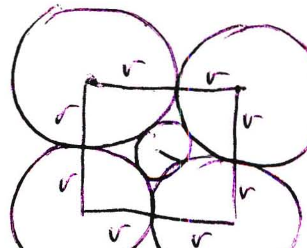
8)



$$A_\Delta = \frac{1}{\cancel{2}}\cdot\cancel{2}\cdot 2\sqrt2 = 2\sqrt2 = \frac{abc}{4R} = \frac{18}{4R}$$
$$R = \frac{18}{8\sqrt2} = \frac{9}{4\sqrt2}$$

Area of the circle $= \pi\frac{81}{32}$

9)



Diagonal of the square could be calculated 2 ways:

$$r+1+1+r = 2r\sqrt2$$
$$2 = r(2\sqrt2-2)$$
$$r = \frac{1}{\sqrt2-1} = \sqrt2+1.$$

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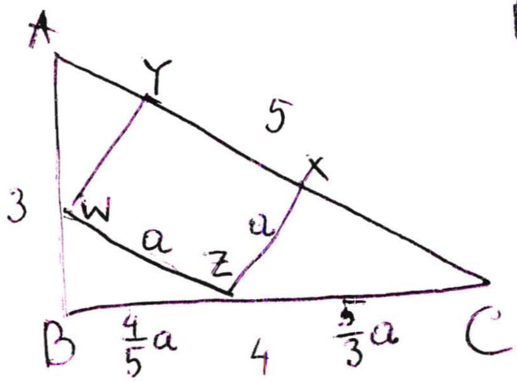

From similarity of triangles we get

$BZ = \frac{4}{5} WZ$

$CZ = \frac{5}{3} XZ$

$\frac{4}{5}a + \frac{5}{3}a = 4 \rightarrow 37a = 60$ so $a = \frac{60}{37}$

11 It just happens that triangle is rightangle triangle, so hypotenuse is twice radius i.e. $c = 6$. Now we have

$a = \frac{3}{5} \cdot 6 = \frac{18}{5}$ $\quad b = \frac{4}{5} \cdot 6 = \frac{24}{5}$ $\quad A = \frac{1}{2} \cdot \frac{18 \cdot 24}{5 \cdot 5} = \frac{216}{25}$

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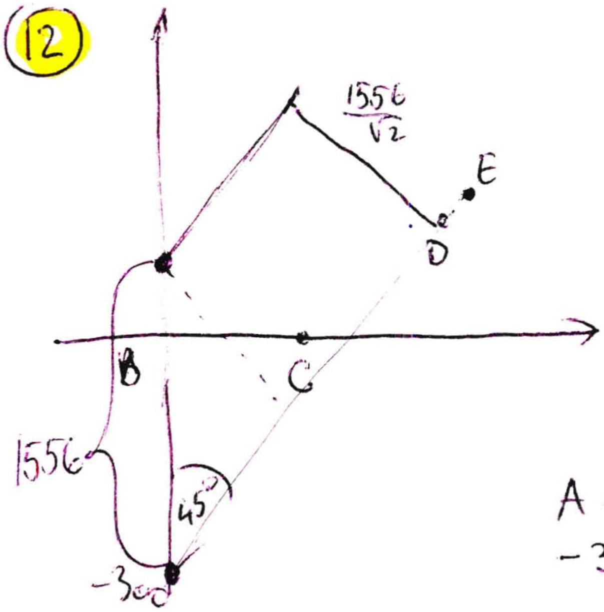

Since $\triangle ABC$ has area 2007 and base 223 the height is $\frac{2007}{223} \cdot 2 = 18$, so A looks like $A(x, \pm 18)$.

D, E belong to the line $y - x = -300$

$A_{\triangle DEA} = \frac{DE \cdot h}{2} = \frac{9\sqrt{2} \cdot h}{2} = 7002$

$h = \frac{1556}{\sqrt{2}}$

A belongs to a line $y - x = S$ where S is -300 ± 1556. Therefore $x = y - S =$
$= y + 300 \pm 1556$

Since $y = \pm 18$, we have $x = \pm 18 + 300 \pm 1556$

when we add all 4 values, $\pm$ cancel out and the sum of the values is $4 \cdot 300 = 1200$.

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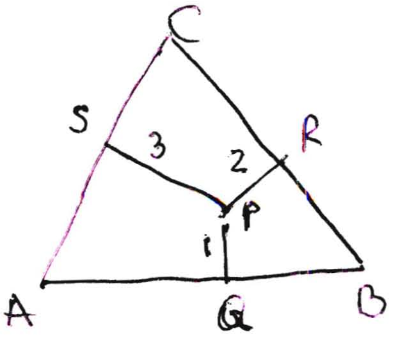

Let a be the side of the triangle.

$\frac{a \cdot 1}{2} + \frac{a \cdot 2}{2} + \frac{a \cdot 3}{2} = \frac{a^2\sqrt{3}}{4}$

$3\cancel{a} = \frac{a^{\cancel{2}}\sqrt{3}}{4} \rightarrow a = 4\sqrt{3}$

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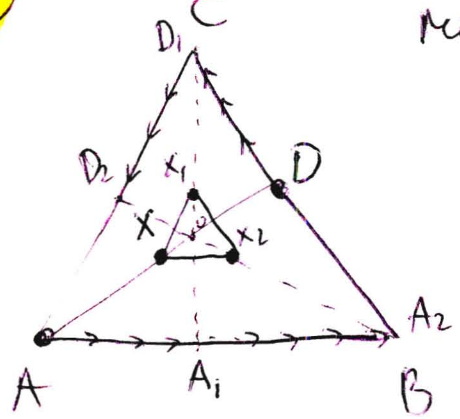
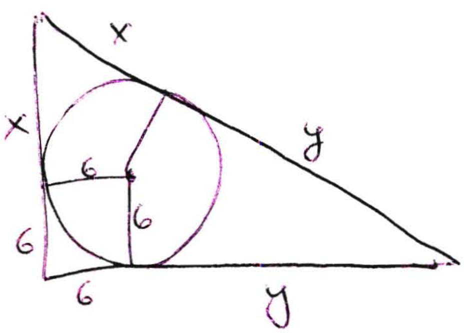


Midpoint traverses ΔXX_1X_2. Since $OA = \frac{2}{3}AD$, $XA = \frac{1}{2}AD$, $OX = \left(-\frac{1}{2}+\frac{2}{3}\right)AD = \frac{1}{6}AD = \frac{1}{4}OA$. Therefore $X_1X_2 = \frac{1}{4}AB$, so Area $\Delta XX_1X_2 = \frac{1}{16}$ Area ΔABC.

15



$A = 3 \cdot P$

$r \cdot s = 6s$

$r = 6$

$(6+x)^2 + (6+y)^2 = (x+y)^2$

$36 + 12x + \cancel{x^2} + 36 + 12y + \cancel{y^2} = \cancel{x^2} + 2xy + \cancel{y^2}$

$36 + 6x + 6y = xy$

$36 = xy - 6x - 6y$

$72 = xy - 6x - 6y + 36$

$72 = (x-6)(y-6) = 1 \cdot 72, 2 \cdot 36, 3 \cdot 24, 4 \cdot 18$
$6 \cdot 12, 8 \cdot 9$

Solving for x, y we get 6 non-congruent triangles.

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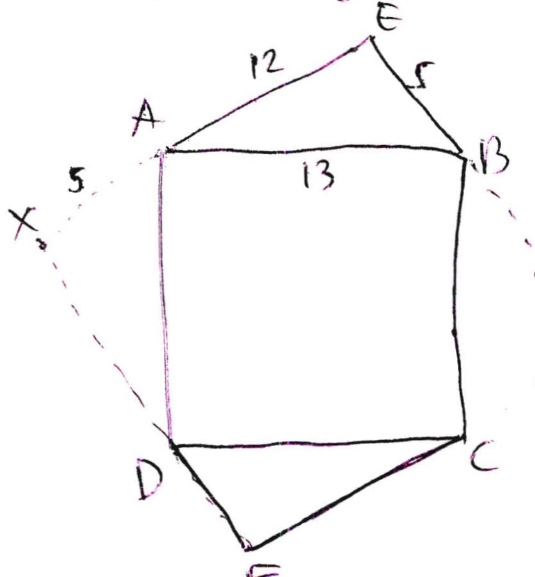

If we extend AE and DF we get another congruent triangle, so $\angle AXD = 90°$.

$\overline{EX} = 17$ is side of a square. Its diagonal $\overline{EF} = 17\sqrt{2}$.

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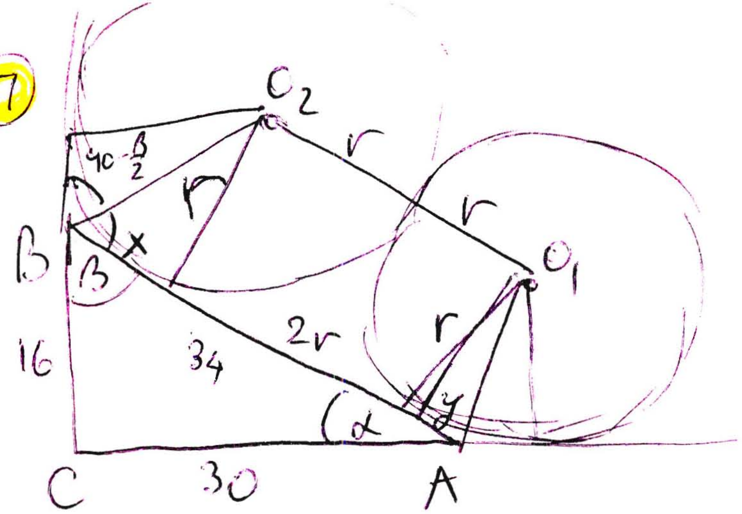

$$\frac{x}{r} = \cot\left(90 - \frac{\beta}{2}\right) = \tan\frac{\beta}{2} = \frac{1-\cos\beta}{\sin\beta} =$$

$$= \frac{1 - \frac{16}{34}}{\frac{30}{34}} = \frac{18}{30} = \frac{3}{5}$$

$$x = \frac{3}{5}r$$

Similarly $\frac{y}{r} = \tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} =$

$$= \frac{1 - \frac{30}{34}}{\frac{16}{34}} = \frac{1}{4} \text{, so } y = \frac{1}{4}r$$

$$x + 2r + y = 34$$

$$\frac{3}{5}r + 2r + \frac{1}{4}r = 34 \quad | \cdot 20$$

$$12r + 40r + 5r = 680$$

$$57r = 680$$

$$r = \frac{680}{57} = \frac{p}{q} \quad \text{so} \quad p+q = 737.$$